RESOLUCIÓN DE PROBLEMAS DE EXAMEN

PROBLEMA-1

Un cono homogéneo de masa m, radio r y semiángulo α rueda por la superficie interior de un cono vertical invertido fijo, de semiángulo $\beta > \alpha$ (Fig. 1). Escribir el lagrangiano del sistema y plantear las ecuaciones de Lagrange.

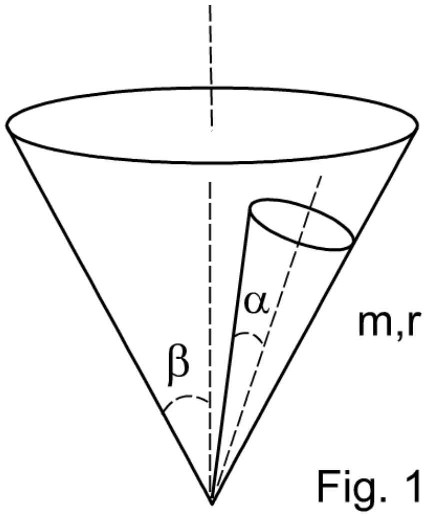
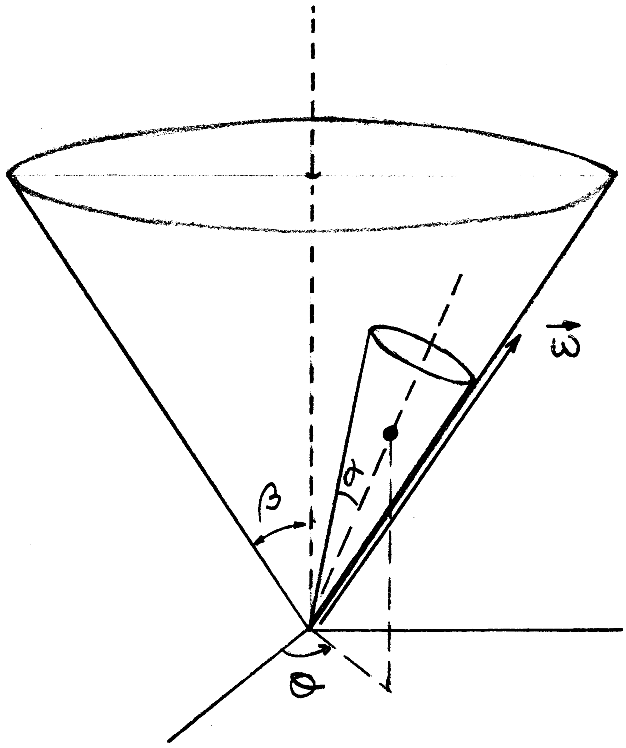


Fig. 1



$\beta > \alpha$

<u>Cono</u> :

h = altura

R = radio

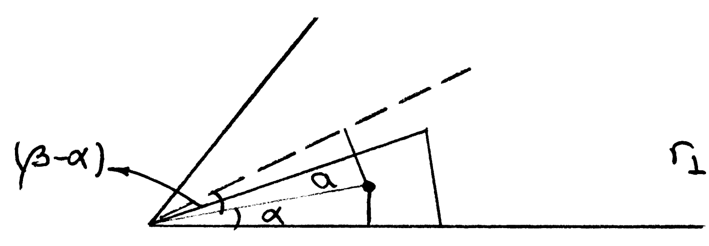
$a = \frac{3}{4}h$ = distancia vértice al C.M.

$L = T - V$

$$\left[\begin{array}{l} T = \frac{1}{2} M \vec{V}_{CM}^{\,2} + \frac{1}{2} \vec{\omega}\, \mathbb{I}_{CM}\, \vec{\omega} \\ V = M g z_{CM} \end{array}\right.$$

$$x_{CM} = a \operatorname{sen}(\beta - \alpha) \cos\theta$$
$$y_{CM} = a \operatorname{sen}(\beta - \alpha) \operatorname{sen}\theta$$
$$z_{CM} = a \cos(\beta - \alpha)$$

$$|\vec{V}_{CM}| = |\vec{\omega} \times \vec{r}\,| = \omega r_\perp = \omega a \operatorname{sen}\alpha$$



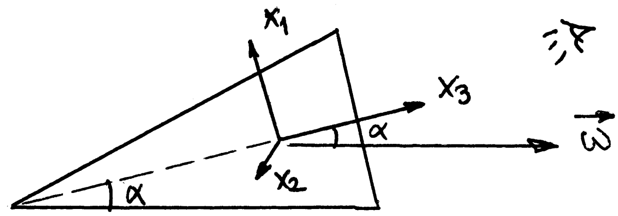
$r_\perp = a \operatorname{sen}\alpha$

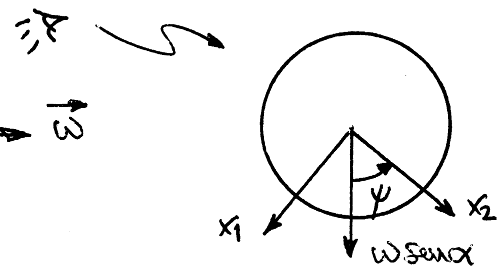
Por otra parte: $\quad V_{CM} = a \operatorname{sen}(\beta - \alpha)\, \dot{\theta}$

Luego:

$$\omega a \operatorname{sen}\alpha = a \dot{\theta} \operatorname{sen}(\beta - \alpha)$$

$$\omega = \dot{\theta}\, \frac{\operatorname{sen}(\beta - \alpha)}{\operatorname{sen}\alpha}$$





$$\omega_3 = \omega \cos\alpha = \dot{\theta}\, \mathrm{cotg}\,\alpha \,\mathrm{sen}(\beta-\alpha)$$

$$\omega_1 = \omega \,\mathrm{sen}\,\alpha \,\mathrm{sen}\,\psi = \dot{\theta}\, \mathrm{sen}(\beta-\alpha)\,\mathrm{sen}\,\psi$$

$$\omega_2 = \omega \,\mathrm{sen}\,\alpha \cos\psi = \dot{\theta}\, \mathrm{sen}(\beta-\alpha)\cos\psi$$

$(I_1 = I_2)$

$$\frac{1}{2}\vec{\omega}\, \mathbb{I}_{cm}\, \vec{\omega} = \frac{1}{2}\left(I_1 \omega_1^2 + I_2 \omega_2^2 + I_3 \omega_3^2\right) =$$

$$= \frac{1}{2} I_1 \left(\omega_1^2 + \omega_2^2\right) + \frac{1}{2} I_3 \omega_3^2 =$$

$$= \frac{1}{2} I_1 \omega^2 \mathrm{sen}^2\alpha + \frac{1}{2} I_3 \omega^2 \cos^2\alpha =$$

$$= \frac{1}{2} I_1 \dot{\theta}^2 \mathrm{sen}^2(\beta-\alpha) + \frac{1}{2} I_3 \dot{\theta}^2 \mathrm{ctg}^2\alpha \,\mathrm{sen}^2(\beta-\alpha)$$

Luego:

$$T = \frac{1}{2} M a^2 \mathrm{sen}^2(\beta-\alpha)\dot{\theta}^2 + \frac{1}{2} I_1 \dot{\theta}^2 \mathrm{sen}^2(\beta-\alpha) +$$
$$+ \frac{1}{2} I_3 \dot{\theta}^2 \mathrm{cotg}^2\alpha \,\mathrm{sen}^2(\beta-\alpha)$$

$$V = Mga\cos(\beta-\alpha) = \mathrm{cte.}$$

$$L = T - V$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}}\right) - \frac{\partial L}{\partial \theta} = 0$$

PROBLEMA-2

Sea el sistema de la Fig.2, constituido por una barra homogénea de masa m y longitud L, acoplada mediante resortes de constantes k_1 y k_2 a un disco homogéneo de masa M y radio R que rueda sobre una superficie vertical. Se supone que la situación representada en la figura constituye la configuración de equilibrio del sistema. a) Considerar la aproximación de pequeñas oscilaciones y escribir el lagrangiano del sistema en términos de m, L, M, R, k_1 y k_2. b) Obtener las matrices $\overline{T}$ y $\overline{V}$.

Considerando los siguientes valores numéricos: $g = 10$ m/s^2, $m = 1$ kg, $M = 2$ kg, $L = 2$ m, $R = 1$ m, k_1= 1 Nw/m, k_2= 2 Nw/m [$I_b = (1/12)mL^2$, $I_d = (1/2)MR^2$], determínense: c) Las frecuencias naturales de oscilación; d) Los modos normales; e) La solución general; f) La ecuación de transformación de coordenadas normales a coordenadas generalizadas.

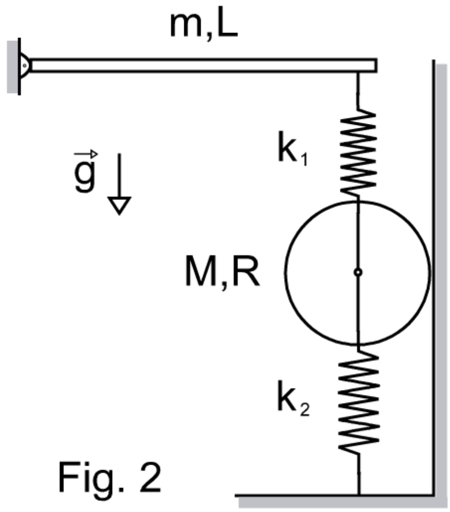
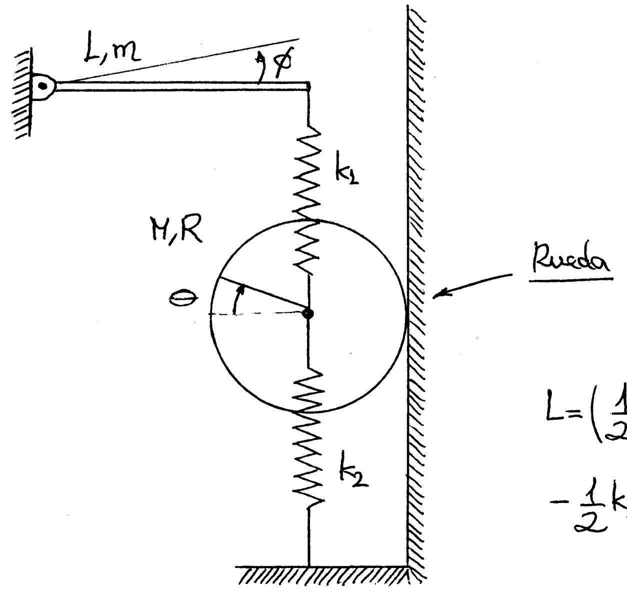


Fig. 2



$$L=\left(\frac{1}{2}MR^2\dot{\theta}^2+\frac{1}{2}I_d\dot{\theta}^2\right)+\left(\frac{1}{2}m\left(\frac{L}{2}\dot{\phi}\right)^2+\frac{1}{2}I_v\dot{\phi}^2\right)-$$
$$-\frac{1}{2}k_1(R\theta-L\operatorname{sen}\phi)^2-\frac{1}{2}k_2R^2\theta^2$$

$$I_d=\frac{1}{2}MR^2 \qquad I_v=\frac{1}{12}mL^2$$

$$L=\frac{1}{2}(MR^2+I_d)\dot{\theta}^2+\frac{1}{2}\left(m\frac{L^2}{4}+I_v\right)\dot{\phi}^2-\frac{1}{2}k_1\left(R^2\theta^2+L^2\operatorname{sen}^2\phi-2RL\theta\operatorname{sen}\phi\right)-\frac{1}{2}k_2R^2\theta^2$$

$$L\simeq\frac{1}{2}(MR^2+I_d)\dot{\theta}^2+\frac{1}{2}\left(m\frac{L^2}{4}+I_v\right)\dot{\phi}^2-\frac{1}{2}k_1\left(R^2\theta^2+L^2\phi^2-2RL\theta\phi\right)-\frac{1}{2}k_2R^2\theta^2$$

$\operatorname{sen}\phi\simeq\phi+\ldots \quad \cos\phi\simeq 1-\frac{\phi^2}{2}+\ldots$

$$\bar{T}=\begin{pmatrix} MR^2+I_d & \\ & m\frac{L^2}{4}+I_v\end{pmatrix} \qquad \bar{V}=\begin{pmatrix}(k_1+k_2)R^2 & -k_1RL\\ -k_1RL & k_1L^2\end{pmatrix} \qquad X\equiv\begin{pmatrix}\theta\\ \phi\end{pmatrix}$$

$M=2$ kg. $\quad m=3$ kg $\quad k_1=2$ Nw-m

$R=1$ m $\quad L=1$ m $\quad k_2=1$ Nw-m

$I_d=\frac{1}{2}MR^2=1$ kg-m²

$I_v=\frac{1}{12}mL^2=\frac{1}{12}3=\frac{1}{4}$ kg-m²

$$\bar{T}=\begin{pmatrix}3 & \\ & 1\end{pmatrix} \qquad \bar{V}=\begin{pmatrix}3 & -2\\ -2 & 2\end{pmatrix}$$

Frecuencias Naturales.

$$0=|\bar{V}-\omega^2\bar{T}|=\begin{vmatrix}3-3\omega^2 & -2\\ -2 & 2-\omega^2\end{vmatrix}=(3-3\omega^2)(2-\omega^2)-4=0$$

$$6-3\omega^2-6\omega^2+3\omega^4-4=0 \qquad (\lambda\equiv\omega^2)$$

$$3\lambda^2-9\lambda+2=0$$

$$\lambda=\frac{9\pm\sqrt{81-24}}{6}=\begin{cases}(+)\ 2.758\\ (-)\ 0.242\end{cases}$$

$\omega_1=1.661\ s^{-1}$

$\omega_2=0.492\ s^{-1}$

Modos Normales

$$(\bar{V}-\omega_i^2 \bar{T})A_i = 0 \; ; \; A_i^T \bar{T} A_i = 1$$

$$\begin{pmatrix} 3-3\omega_i^2 & -2 \\ -2 & 2-\omega_i^2 \end{pmatrix} \begin{pmatrix} a_{1i} \\ a_{2i} \end{pmatrix} = 0 \longrightarrow \begin{cases} (3-3\omega_i^2)\, a_{1i} - 2a_{2i} = 0 \\ -2a_{1i} + (2-\omega_i^2)\, a_{2i} = 0 \end{cases}$$

$$3a_{1i}^2 + a_{2i}^2 = 1$$

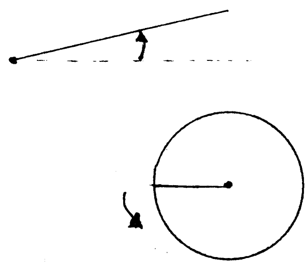
$$a_{1i} = \frac{1}{2}(2-\omega_i^2)\, a_{2i} \longrightarrow \left[3\,\frac{1}{4}(2-\omega_i^2)^2 + 1\right] a_{2i}^2 = 1$$

$$a_{2i} = \left[\frac{3}{4}(2-\omega_i^2)^2 + 1\right]^{-1/2}$$

Modo 1: $\omega_1^2 = 2.758$

$$a_{21} = 0.836 \qquad a_{11} = -0.379 \times 0.836 = -0.317 \longrightarrow A_1 = \begin{pmatrix} -0.317 \\ 0.836 \end{pmatrix}$$

$$\begin{cases} \theta = -0.317\, |c_1| \cos(\omega_1 t + \phi_1) \\ \phi = 0.836\, |c_1| \cos(\omega_1 t + \phi_1) \end{cases}$$



Modo 2, $\omega_2^2 = 0.242$

$$a_{22} = 0.549 \longrightarrow a_{12} = 0.483 \longrightarrow A_2 = \begin{pmatrix} 0.483 \\ 0.549 \end{pmatrix}$$

$$\begin{cases} \theta = 0.483\, |c_2| \cos(\omega_2 t + \phi_2) \\ \phi = 0.549\, |c_2| \cos(\omega_2 t + \phi_2) \end{cases}$$

Solución General

$$X = b_1 X_1 + b_2 X_2$$

$$\theta(t) = -0.317\, c_1' \cos(\omega_1 t + \phi_1) + 0.483\, c_2' \cos(\omega_2 t + \phi_2)$$

$$\phi(t) = 0.836\, c_1' \cos(\omega_1 t + \phi_1) + 0.549\, c_2' \cos(\omega_2 t + \phi_2)$$

Cambio de Coordenadas

$$\begin{pmatrix} \theta \\ \phi \end{pmatrix} = \begin{pmatrix} -0.317 & 0.483 \\ 0.836 & 0.549 \end{pmatrix} \begin{pmatrix} q_1 \\ q_2 \end{pmatrix}$$