Probability distribution of arrival times in quantum mechanics

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(Received 18 September 1997)

In a previous paper [V. Delgado and J. G. Muga, Phys. Rev. A **56**, 3425 (1997)] we introduced a self-adjoint operator $\hat{\mathcal{T}}(X)$ whose eigenstates can be used to define consistently a probability distribution of the time of arrival at a given spatial point. In the present work we show that the probability distribution previously proposed can be well understood on classical grounds in the sense that it is given by the expectation value of a certain positive-definite operator $\hat{J}^{(+)}(X)$, which is nothing but a straightforward quantum version of the modulus of the classical current. For quantum states highly localized in momentum space about a certain momentum $p_0 \neq 0$, the expectation value of $\hat{J}^{(+)}(X)$ becomes indistinguishable from the quantum probability current. This fact may provide a justification for the common practice of using the latter quantity as a probability distribution of arrival times. [S1050-2947(98)03602-6]

PACS number(s): 03.65.Bz

I. INTRODUCTION

Standard quantum mechanics is mainly concerned with probability distributions of measurable quantities at a given instant of time. Such distributions can be inferred from the formalism in terms of projections of the instantaneous state vector $|\psi(t)\rangle$ onto appropriate subspaces of the whole Hilbert space of physical states. However, one may also be interested in the probability that a certain physical quantity takes a definite value between the instants of time t and $t+dt$. Let us assume this quantity to be the position of a particle and restrict ourselves to one spatial dimension. What is the probability distribution of arrival times at a detector situated at a given point $x=X$? Standard quantum theory is based on the assumption that, by reducing sufficiently the experimental uncertainties involved, the outcomes of any measurement process will reproduce, within any desirable precision, the ideal distribution inferred from the spectral decomposition of a certain self-adjoint operator associated with the physical quantity under consideration. It is therefore implicitly assumed that the quantum formalism can provide a prediction for the experimental results without having to make reference to the specific properties of the measuring device involved. Since the distribution of arrival times at a given spatial point is, in principle, a measurable quantity that can be determined via a time-of-flight experiment, it is reasonable to ask for an apparatus-independent theoretical prediction.

In classical statistical mechanics the above question has a definite answer: The (unnormalized) probability distribution of arrival times at X for a certain statistical ensemble of particles of mass m, moving along a well-defined spatial direction (i.e., either with momenta $p>0$ or with momenta $p<0$), is given by the average current at X,

$$\langle J(X)\rangle = \int \int f(x,p,t)\,\frac{p}{m}\,\delta(x-X)\,dx\;dp, \tag{1}$$

where $f(x,p,t)$ represents the phase-space distribution function characterizing the statistical ensemble. In quantum mechanics, however, things turn out to be much more involved. In particular, a straightforward application of the correspondence principle would lead us to consider the expectation value of the current operator

$$\hat{J}(X) = \frac{1}{2m}\left(\hat{P}|X\rangle\langle X| + |X\rangle\langle X|\hat{P}\right) \tag{2}$$

($\hat{P}$ denoting the momentum operator) as the most natural quantum candidate for the probability distribution of the time of arrival at a point X. However, even though such a definition has been widely used in recent times [1–4], it cannot be considered as a satisfactory solution because of the fact that the expectation value of $\hat{J}(X)$ is not positive definite, even for wave packets containing only positive-momentum components. Nonetheless, when quantum backflow contributions become negligible, one expects the expectation value of the current operator to be a good approximation to the actual probability distribution of arrival times.

The difficulty for defining such probability distributions is nothing but a mere aspect of the more fundamental problem of the nonexistence of a quantum time operator conjugate to the Hamiltonian. The reason for this latter fact lies, basically, in the incompatibility of such a time operator with the semibounded nature of the Hamiltonian spectrum [5–7].

In spite of detailed work by Allcock [6] denying the possibility of incorporating the time-of-arrival concept in the quantum framework, more recently there has been considerable effort in defining a probability distribution of the time of arrival of a quantum particle at a given spatial point [1–4,7–12]. The incorporation of such probability distributions in the formalism of quantum mechanics has both conceptual and practical interest. In particular, this issue is closely related to the problem of the temporal characterization of tunneling (the so-called tunneling time problem) [13–21], whose understanding is important for its possible application in semiconductor technology.

In Ref. [1], Dumont and Marchioro, primarily concerned with tunneling-time distributions, proposed the probability current as a quantum definition of the (unnormalized) probability distribution of arrival times at a point sufficiently far to the right of a one-dimensional potential barrier. Leavens [2] has shown that this result can also be derived within Bohm's trajectory interpretation of quantum mechanics by making the assumption that particles are not reflected back through the point X [i.e., $\langle\psi(t)|\hat{J}(X)|\psi(t)\rangle<0$ does not occur for any t]. On the other hand, Muga *et al.* [4] have provided an operational justification of such a definition by simulating the detection of incoming particles by a destructive procedure. More recently, Grot *et al.* [9] have faced the problem from a somewhat different perspective. These authors construct a suitable self-adjoint operator in order to infer a probability distribution of arrival times from its spectral decomposition. More specifically, starting from the classical equations of motion for a particle moving freely in one spatial dimension and solving for the time, they arrive at the operator

$$\hat{T}(X)=\sqrt{\frac{m}{\hat{P}_{\rm H}(0)}}[X-\hat{X}_{\rm H}(0)]\sqrt{\frac{m}{\hat{P}_{\rm H}(0)}} \tag{3}$$

as a natural candidate for the time of arrival of a quantum free particle at the spatial point X. In the above equation $\hat{X}_{\rm H}(t)$ and $\hat{P}_{\rm H}(t)$ denote the position and momentum operators in the Heisenberg picture, respectively, and are related to the corresponding Schrödinger operators by

$$\hat{O}_{\rm H}(t)=e^{i\hat{H}_0t/\hbar}\hat{O}e^{-i\hat{H}_0t/\hbar}, \tag{4}$$

where $\hat{O}$ stands for $\hat{X}$ or $\hat{P}$, and $\hat{H}_0=\hat{P}^2/2m$ is the Hamiltonian of the free particle. The operator (3) has the interest that it represents a quantum version of the corresponding classical expression $t(X)=m[X-x(0)]/p(0)$ obtained by straightforward application of the correspondence principle and the *canonical quantization method* [22], which states that classical equations remain formally valid in the quantum framework provided that one makes the substitution of Poisson brackets by commutators $\{\ ,\ \}\to 1/i\hbar[\ ,\ ]$ and interprets the classical dynamical variables as self-adjoint operators in the Heisenberg picture. Of course, whenever the classical expression under consideration contains products of dynamical variables having a nonvanishing Poisson bracket (as is the case for position and momentum), the mere application of the canonical quantization method does not guarantee the unambiguous construction of the corresponding quantum quantity. In fact, a specific symmetrization or quantization rule has been explicitly chosen in Eq. (3). For instance, another possible quantum operator obtained from the same classical expression via a different symmetrization rule (which has been previously introduced by Aharonov and Bohm [23]) is

$$\hat{T}(X)=\frac{1}{2}\left([X-\hat{X}_{\rm H}(0)]\frac{m}{\hat{P}_{\rm H}(0)}+\frac{m}{\hat{P}_{\rm H}(0)}[X-\hat{X}_{\rm H}(0)]\right). \tag{5}$$

Unfortunately, despite the fact that the above two operators are exactly what one would expect by virtue of the correspondence principle, none of them is self-adjoint. To circumvent this difficulty, Grot *et al.* proposed a modified time operator such that, when acting on states with no zero-momentum components, it leads to the same results as the operator (3) defined above.

In a previous paper [7] we followed a different route: Guided by the fact that, in general, a self-adjoint time operator conjugate to the Hamiltonian does not exist, we instead looked for a self-adjoint operator $\hat{\mathcal{T}}(X)$ with dimensions of time, conjugate to a conveniently defined energy operator having a nonbounded spectrum. We showed that the orthogonal spectral decomposition of such an operator can be used to define consistently a probability distribution of arrival times at a given spatial point within the standard formalism of quantum mechanics.

In this paper we are mainly interested in relating the formulation proposed in Ref. [7] with the corresponding classical formulation. A quantum expression looks more natural as long as it is possible to derive it from a known classical quantity by applying certain specific quantization rules (even though such a procedure is by no means a necessary condition for the validity of a quantum formulation). We begin by briefly reviewing the relevant formalism in Sec. II. In Sec. III we consider the semiclassical limit of the proposed probability distribution of arrival times. In Sec. IV we show that such a probability distribution can be well understood on classical grounds in the sense that it is formally analogous to its corresponding classical counterpart. In this section we also provide a relation between the expectation value of the self-adjoint operator $\hat{\mathcal{T}}(X)$ and the expectation values of the operators $\hat{T}(X)$ given by Eqs. (3) and (5). In Sec. V we analyze under what circumstances the proposed probability distribution can be replaced, to a good approximation, by the probability current, which has been frequently used, in practice, as a quantum probability distribution of arrival times. Finally, we conclude in Sec. VI.

II. FORMALISM

In looking for a probability distribution of arrival times within the framework of standard quantum mechanics, we introduced in Ref. [7] a self-adjoint energy operator $\hat{\mathcal{H}}$ defined by

$$\hat{\mathcal{H}}\equiv{\rm sgn}(\hat{P})\frac{\hat{P}^2}{2m}, \tag{6}$$

where, in terms of a basis of momentum eigenstates $\{|p\rangle\}$, the operator ${\rm sgn}(\hat{P})$ reads

$${\rm sgn}(\hat{P})\equiv\int_0^\infty dp(|p\rangle\langle p|-|-p\rangle\langle -p|). \tag{7}$$

The normalization has been chosen so that the states $|p\rangle$ satisfy the closure and orthonormalization relations

$$\int_{-\infty}^{+\infty}dp|p\rangle\langle p|=\mathbf{1}, \tag{8}$$

$$\langle p|p'\rangle=\delta(p-p'). \tag{9}$$

The motivation for introducing the operator $\hat{\mathcal{H}}$, which essentially represents the energy of the free particle with the sign of its momentum, lies in the fact that, unlike the Hamiltonian, it exhibits a nonbounded spectrum. It is therefore possible to define a self-adjoint operator with dimensions of time $\hat{\mathcal{T}}(X)$ by simply demanding it to be conjugate to $\hat{\mathcal{H}}$, i.e.,

$$[\hat{\mathcal{H}},\hat{\mathcal{T}}(X)]=e^{-i\hat{P}X/\hbar}[\hat{\mathcal{H}},\hat{\mathcal{T}}(0)]e^{+i\hat{P}X/\hbar}=i\hbar. \tag{10}$$

This procedure led us to an operator $\hat{\mathcal{T}}(X)$ whose orthogonal spectral decomposition reads

$$\hat{\mathcal{T}}(X)=\int_{-\infty}^{+\infty}d\tau\tau|\tau;X\rangle\langle\tau;X|, \tag{11}$$

where the states $|\tau;X\rangle$, which constitute a complete and orthogonal set, are given by

$$|\tau;X\rangle=h^{-1/2}\int_{-\infty}^{+\infty}dp\,\sqrt{\frac{|p|}{m}}e^{i[\mathrm{sgn}(p)(p^2/2m)\tau-pX]/\hbar}|p\rangle. \tag{12}$$

In order to facilitate an interpretation in terms of measurement results, it turns out to be most convenient to decompose the eigenstates $|\tau;X\rangle$ as a superposition of negative- and positive-momentum contributions, in the form

$$|\tau;X\rangle=|t=-\tau,-;X\rangle+|t=+\tau,+;X\rangle, \tag{13}$$

with $|t,\pm;X\rangle$ defined by

$$|t,\pm;X\rangle=h^{-1/2}\int_0^{\infty}dp\,\sqrt{\frac{p}{m}}e^{i[(p^2/2m)t\mp pX]/\hbar}|\pm p\rangle. \tag{14}$$

As can be easily verified, even though the states $|t,\pm;X\rangle$ constitute a complete set they are not orthogonal. Specifically,

$$\sum_{\alpha=\pm}\int_{-\infty}^{+\infty}dt|t,\alpha;X\rangle\langle t,\alpha;X|=\mathbf{1}, \tag{15}$$

$$\langle t,\alpha|t',\alpha'\rangle=\frac{1}{2}\delta_{\alpha\alpha'}\left(\delta(t-t')-P\frac{i}{\pi(t-t')}\right). \tag{16}$$

Despite this fact, the decomposition (13) is interesting because the variable t, unlike τ, admits a proper interpretation as a physical time. In particular, the states $|t,\pm;X\rangle$ not only have the desirable time-translation property

$$e^{i\hat{H}_0t'/\hbar}|t,\pm;X\rangle=|t+t',\pm;X\rangle, \tag{17}$$

but also transform under time reversal as $|t,\pm\rangle\rightarrow|-t,\mp\rangle$.

Consider a free particle propagating along the x axis toward a detector located at a given point X. We shall assume that its actual state at $t=0$ is, in the position representation, either a linear superposition of positive plane waves $|\psi_+(0)\rangle$ (corresponding to particles arriving at the detector from the left) or a linear superposition of negative plane waves $|\psi_-(0)\rangle$ (corresponding to particles arriving at the detector from the right). At any instant of time the state vectors $|\psi_\pm(t)\rangle$ satisfy the identity

$$|\psi_\pm(t)\rangle\equiv\Theta(\pm\hat{P})|\psi_\pm(t)\rangle, \tag{18}$$

where $\Theta(\pm\hat{P})$ are projectors onto the subspaces spanned by plane waves either with positive or with negative momenta

$$\Theta(\pm\hat{P})=\int_0^{\infty}dp|\pm p\rangle\langle\pm p|. \tag{19}$$

It can be shown that for normalizable states satisfying Eq. (18) and vanishing (in momentum representation) faster than p as $p\rightarrow0$, it holds that [7]

$$\pm\langle\psi_\pm|\hat{\mathcal{T}}(X)|\psi_\pm\rangle=\int_{-\infty}^{+\infty}d\tau\tau\langle\psi_\pm|\pm\tau;X\rangle\langle\pm\tau;X|\psi_\pm\rangle$$
$$=\frac{\int_{-\infty}^{+\infty}d\tau\tau\langle\psi_\pm(\tau)|\hat{J}(X)|\psi_\pm(\tau)\rangle}{\int_{-\infty}^{+\infty}d\tau\langle\psi_\pm(\tau)|\hat{J}(X)|\psi_\pm(\tau)\rangle}, \tag{20}$$

where use has been made of Eq. (11) and $|\psi_\pm\rangle$ denotes the state of the particle in the Heisenberg picture,

$$|\psi_\pm\rangle=e^{i\hat{H}_0\tau/\hbar}|\psi_\pm(\tau)\rangle=|\psi_\pm(0)\rangle. \tag{21}$$

As already stated, the right-hand side of Eq. (20) can be recognized as a quantum version of the mean arrival time at X, obtained by straightforward application of the correspondence principle to the analogous classical expression for a statistical ensemble of particles propagating along a well-defined spatial direction. Furthermore, the positive-definite quantity $\langle\psi_\pm|\pm\tau;X\rangle\langle\pm\tau;X|\psi_\pm\rangle$ satisfies

$$\int_{-\infty}^{+\infty}d\tau\langle\psi_\pm|\pm\tau;X\rangle\langle\pm\tau;X|\psi_\pm\rangle=\langle\psi_\pm|\psi_\pm\rangle=1. \tag{22}$$

Therefore, in a quantum framework, the mean arrival time at X can be defined consistently by

$$\langle t_X\rangle_\pm=\pm\langle\psi_\pm|\hat{\mathcal{T}}(X)|\psi_\pm\rangle$$
$$=\int_{-\infty}^{+\infty}d\tau\tau\langle\psi_\pm|\pm\tau;X\rangle\langle\pm\tau;X|\psi_\pm\rangle. \tag{23}$$

Accordingly, the probability amplitude $\Psi_\pm(t=\tau;X)$ of arriving at X at the instant $t=\tau$, coming from the left (+) or right (−), would be given by

$$\Psi_\pm(t=\tau;X)\equiv\langle\pm\tau;X|\psi_\pm\rangle=\langle t=\pm\tau,\pm;X|\psi_\pm\rangle$$
$$=h^{-1/2}\int_0^{\infty}dp\,\sqrt{\frac{p}{m}}\langle\pm p|\psi_\pm\rangle e^{-i[(p^2/2m)\tau\mp pX]/\hbar} \tag{24}$$

and the corresponding probability density takes the form

$$|\Psi_{\pm}(t=\tau;X)|^2=\int_0^\infty dp'\int_0^\infty dp\,\frac{\sqrt{p\,p'}}{m\,h}\langle\psi_\pm|\pm p\rangle\times\langle\pm p'|\psi_\pm\rangle e^{i(p^2/2m-p'^2/2m)\tau/\hbar}e^{\mp i(p-p')X/\hbar}. \tag{25}$$

The above formulation can be generalized in order to define a probability distribution of arrival times at an asymptotic point X behind a one-dimensional potential barrier. Indeed, provided that the potential $V(x)$ vanishes sufficiently fast, far away from the scattering center, as to guarantee the validity of the standard scattering formalism, it can be shown that Eqs. (22)–(25) remain formally valid with the only substitution

$$|\psi_+\rangle\rightarrow\frac{|\psi_{\rm tr}\rangle}{\sqrt{\langle\psi_{\rm tr}|\psi_{\rm tr}\rangle}}, \tag{26}$$

where the (unnormalized) freely evolving transmitted state $|\psi_{\rm tr}\rangle$ can be written in terms of the scattering operator $\hat{S}$ as

$$|\psi_{\rm tr}\rangle=\Theta(\hat{P})\hat{S}|\psi_{\rm in}\rangle=\int_0^\infty dp\;T(p)\langle p|\psi_{\rm in}\rangle|p\rangle. \tag{27}$$

In the above equation $T(p)$ denotes the transmission coefficient characterizing the potential barrier, and the state vector $|\psi_{\rm in}\rangle$ [which is assumed to satisfy the identity $|\psi_{\rm in}\rangle\equiv\Theta(\hat{P})|\psi_{\rm in}\rangle$] represents the incoming asymptote of the actual scattering state of the particle at $t=0$. Whenever this latter state $|\psi(0)\rangle$ does not overlap appreciably with the potential barrier, it becomes physically indistinguishable from $|\psi_{\rm in}\rangle$ and, consequently, it is not necessary to discriminate between them in practice [24].

Since the presence of a potential barrier is not relevant for our purposes in this work, in what follows we shall restrict ourselves to the free case. More specifically, we shall consider a freely moving particle characterized by a state vector $|\psi_\pm(t)\rangle$ satisfying Eq. (18). Nonetheless, this assumption does not imply any loss of generality in practice since the formulation below can be systematically generalized by means of the substitution (26).

III. SEMICLASSICAL LIMIT

In spite of the fact that the expectation value

$$\pm\langle\hat{J}(X)\rangle_\pm\equiv\pm\langle\psi_\pm(\tau)|\hat{J}(X)|\psi_\pm(\tau)\rangle=\pm\langle\psi_\pm|\hat{J}_{\rm H}(X,\tau)|\psi_\pm\rangle \tag{28}$$

cannot be properly considered as a probability density of arrival times, it represents, however, a natural quantum version of the corresponding classical probability density. For this reason it is instructive to investigate the connection between such an expectation value and the quantity $\langle\psi_\pm|\pm\tau;X\rangle\langle\pm\tau;X|\psi_\pm\rangle$, which, as stated above, can be interpreted consistently as a quantum probability density of arrival times. To this end, by inserting twice the resolution of unity in terms of a momentum basis, we write

$$\pm\langle\psi_\pm|\hat{J}_{\rm H}(X,\tau)|\psi_\pm\rangle=\int_0^\infty dp\int_0^\infty dp'\langle\psi_\pm|\pm p\rangle\langle\pm p'|\psi_\pm\rangle\times\left(\frac{p+p'}{2mh}\right)e^{i(p^2/2m-p'^2/2m)\tau/\hbar}\times e^{\mp i(p-p')X/\hbar}. \tag{29}$$

Expressing next the probability amplitude $\langle\pm p|\psi_\pm\rangle$ in polar form as

$$\langle\pm p|\psi_\pm\rangle=|\langle\pm p|\psi_\pm\rangle|e^{i\phi_\pm(p)/\hbar} \tag{30}$$

and introducing the functional

$$I_\pm[f]\equiv\int_0^\infty dp\;f(p)|\langle\pm p|\psi_\pm\rangle|e^{i\chi_\pm(p)/\hbar}, \tag{31}$$

where the phase $\chi_\pm(p)$ is defined as

$$\chi_\pm(p)\equiv\phi_\pm(p)-\frac{p^2}{2m}\tau\pm pX, \tag{32}$$

we can finally rewrite Eq. (29) in the form

$$\pm\langle\psi_\pm|\hat{J}_{\rm H}(X,\tau)|\psi_\pm\rangle=\frac{1}{mh}\frac{1}{2}(I_\pm^*[p]I_\pm[1]+\text{c.c.}). \tag{33}$$

On the other hand, the probability density (25) (which, unlike the probability current, is manifestly positive definite) can be written in a completely analogous manner as

$$\langle\psi_\pm|\pm\tau;X\rangle\langle\pm\tau;X|\psi_\pm\rangle=\frac{1}{mh}(I_\pm^*[\sqrt{p}]I_\pm[\sqrt{p}]). \tag{34}$$

The two expressions (33) and (34) are especially suitable for investigating the semiclassical limit $\hbar\rightarrow0$. Indeed, in this limit the asymptotic expansion of the Fourier-type integral (31) is given, to leading order, by [25]

$$I_\pm[f]\sim e^{i(\pi/4)\,\text{sgn}[\chi''_\pm(p_0)]}e^{i\chi_\pm(p_0)/\hbar}f(p_0)\times|\langle\pm p_0|\psi_\pm\rangle|\sqrt{\frac{h}{|\chi''_\pm(p_0)|}}, \tag{35}$$

where p_0 is the stationary point of the phase $\chi_\pm(p)$, defined implicitly by

$$\chi'_\pm(p_0)\equiv\phi'_\pm(p_0)-\frac{p_0}{m}\tau\pm X=0, \tag{36}$$

and primes are used to denote differentiation with respect to the momentum p.

By substituting Eq. (35) into Eqs. (33) and (34) we obtain, to leading order as $\hbar\rightarrow0$,

$$\langle\psi_\pm|\pm\tau;X\rangle\langle\pm\tau;X|\psi_\pm\rangle\sim\pm\langle\psi_\pm|\hat{J}_{\rm H}(X,\tau)|\psi_\pm\rangle\sim\frac{p_0}{m}|\psi_\pm(X,\tau)|^2. \tag{37}$$

In the last step of the above formula we have used that

$$\psi_{\pm}(X,\tau)=\langle X|e^{i(\hat{P}^2/2m)\tau/\hbar}|\psi_{\pm}\rangle=h^{-1/2}I_{\pm}[1], \tag{38}$$

so that, as $\hbar\to 0$, we have

$$|\psi_{\pm}(X,\tau)|^2\sim\frac{|\langle\pm p_0|\psi_{\pm}\rangle|^2}{|\chi''_{\pm}(p_0)|}. \tag{39}$$

The information contained in Eq. (37) represents the main result of this section. This equation reflects that the proposed probability density of arrival times $|\langle\pm\tau;X|\psi_{\pm}\rangle|^2$ coincides, in the semiclassical limit, with the quantum probability current $\pm\langle\psi_{\pm}|\hat{J}_{\mathrm{H}}(X,\tau)|\psi_{\pm}\rangle$, which in turn is given in this limit by the product of the probability density $|\psi_{\pm}(X,\tau)|^2$ and the group velocity p_0/m (and, consequently, becomes a positive quantity). This fact suggests that the probability density defined in Eq. (25) is nothing but a quantum version of the modulus of the classical average current $\langle J(X)\rangle$, which, as already stated, plays the role of a probability distribution of arrival times at X for a classical statistical ensemble of particles moving along a well-defined spatial direction. In the next section we shall see that this is indeed the case.

IV. POSITIVE-DEFINITE CURRENT

Let us concentrate on Eq. (12), which defines the eigenstates of the operator $\hat{\mathcal{T}}(X)$ in terms of a basis of momentum eigenstates. By introducing the self-adjoint operator

$$\sqrt{|\hat{P}|}\equiv\int_{-\infty}^{+\infty}dp\,\sqrt{|p|}|p\rangle\langle p|, \tag{40}$$

we can express $|\tau;X\rangle$ in an alternative form that exhibits no explicit dependence on any particular representation and proves to be most convenient for our purposes. Indeed,

$$\begin{aligned}|\tau;X\rangle&=\int_{-\infty}^{+\infty}dp\left\langle p\left|\sqrt{\frac{|\hat{P}|}{m}}e^{i\,\mathrm{sgn}(\hat{P})(\hat{P}^2/2m)\tau/\hbar}\right|X\right\rangle|p\rangle\\&=\sqrt{\frac{|\hat{P}|}{m}}e^{i\,\mathrm{sgn}(\hat{P})(\hat{P}^2/2m)\tau/\hbar}|X\rangle.\end{aligned} \tag{41}$$

Correspondingly, the probability amplitude for particle detection at the spatial point X [coming from the left $(+)$ or right $(-)$] at time $t=\tau$ takes the form [Eq. (24)]

$$\begin{aligned}\Psi_{\pm}(t=\tau;X)&=\left\langle X\left|\sqrt{\frac{|\hat{P}|}{m}}e^{-i(\hat{P}^2/2m)\tau/\hbar}\right|\psi_{\pm}\right\rangle\\&=\left\langle X\left|\sqrt{\frac{|\hat{P}|}{m}}\right|\psi_{\pm}(\tau)\right\rangle.\end{aligned} \tag{42}$$

This equation shows that the probability amplitude of arriving at X at time τ is nothing but the probability amplitude of finding the state $\sqrt{|\hat{P}|/m}|X\rangle$ in the (Schrödinger) state vector $|\psi_{\pm}(\tau)\rangle$ characterizing the particle dynamics at $t=\tau$. On the other hand, the corresponding probability density reads

$$\begin{aligned}P_X^{(\pm)}(\tau)|_{\mathrm{quant}}&=|\Psi_{\pm}(t=\tau;X)|^2\\&=\left\langle\psi_{\pm}(\tau)\left|\sqrt{\frac{|\hat{P}|}{m}}\delta(\hat{X}-X)\sqrt{\frac{|\hat{P}|}{m}}\right|\psi_{\pm}(\tau)\right\rangle,\end{aligned} \tag{43}$$

where use has been made of the identity $|X\rangle\langle X|\equiv\delta(\hat{X}-X)$.

Before proceeding further it is convenient to consider the *normalized* probability distribution of arrival times at the point X for a classical statistical ensemble of *free* particles of mass m, coming either from the left $(p>0)$ or from the right $(p<0)$. Such a probability distribution can be obtained from Eq. (1). Indeed, by defining $J^{(+)}(X)$ as the modulus of the classical current

$$J^{(+)}(X)\equiv|J(X)|=|p|/m\,\delta(x-X) \tag{44}$$

and using a convenient notation, it takes the form

$$\begin{aligned}P_X^{(\pm)}(\tau)|_{\mathrm{class}}&=\pm\langle J(X)\rangle_{\pm}\\&=\int\int f_{\pm}(x,p,\tau)\frac{|p|}{m}\delta(x-X)dx\,dp\\&=\langle J^{(+)}(X)\rangle_{\pm},\end{aligned} \tag{45}$$

where the phase-space distribution function satisfies the identity

$$f_{\pm}(x,p,t)\equiv\Theta(\pm p)f_{\pm}(x,p,t) \tag{46}$$

and the modulus in the integrand of Eq. (45) comes from the normalization factor (which, in the free case, takes the value ± 1).

A comparison between expressions (43) and (45) shows that the quantum probability density of arrival times defined above [Eq. (43)] can be considered as a quantum version of the corresponding classical expression, obtained by associating to the average of the classical positive current $J^{(+)}(X)\equiv|J(X)|$ the expectation value of the positive definite current operator $\hat{J}^{(+)}(X)$

$$J^{(+)}(X)\equiv\frac{|p|}{m}\delta(x-X)\to\hat{J}^{(+)}(X)\equiv\sqrt{\frac{|\hat{P}|}{m}}\delta(\hat{X}-X)\sqrt{\frac{|\hat{P}|}{m}}. \tag{47}$$

It should be noted, however, that the relation existing between the classical current $J(X)$ and the corresponding quantum operator $\hat{J}(X)$ [given by Eq. (2)] is somehow different from that existing between $J^{(+)}(X)$ and $\hat{J}^{(+)}(X)$. Indeed, $\hat{J}(X)$ can also be considered as the quantum operator corresponding to the classical current $J(X)$ by virtue of the Weyl-Wigner quantization rule, whereas the same does not hold true for the positive current defined above.

The Weyl-Wigner quantization rule is a mapping that associates with every phase-space function $g(x,p)$ a quantum operator $\hat{G}(\hat{X},\hat{P})$ with an expectation value satisfying

$$\langle\hat{G}(\hat{X},\hat{P})\rangle=\int\int f_{\mathrm{W}}(x,p)g(x,p)dx\,dp, \tag{48}$$

where the *Wigner function* $f_{\mathrm{W}}(x,p)$ plays the role of a quasiprobability distribution function in phase space [26] and can be expressed in terms of the quantum density operator $\hat{\rho}$ characterizing the physical system as [27]

$$f_{\mathrm{W}}(x,p)=\frac{1}{4\pi^2}\int\int\int\left\langle q+\frac{\tau\hbar}{2}|\hat{\rho}|q-\frac{\tau\hbar}{2}\right\rangle \times e^{-i[\theta(x-q)+\tau p]}d\theta\, d\tau\, dq. \tag{49}$$

By substituting Eq. (49) into Eq. (48) it can be shown that $\hat{G}(\hat{X},\hat{P})$ is given by

$$\hat{G}(\hat{X},\hat{P})=\frac{1}{4\pi^2}\int\int\int\int g(x,p) \times e^{i[\theta(\hat{X}-q)+\tau(\hat{P}-p)]}dx\; dp\; d\theta\; d\tau, \tag{50}$$

and taking $g(x,p)\equiv p/m\,\delta(x-X)$ in the integrand of Eq. (50) one arrives, after some algebra, at the current operator $\hat{J}(X)$ defined by Eq. (2). Even though a similar relation does not exist for $J^{(+)}(X)\equiv|J(X)|$ it still holds true that the positive-definite operator $\hat{J}^{(+)}(X)$ represents a natural quantum version of the modulus of the classical current. Accordingly, for free particles propagating along a well-defined spatial direction the probability density of the time of arrival at a given point X at time $t=\tau$ can be defined consistently, within both a classical and a quantum-mechanical framework, as the instantaneous mean value of the modulus of the current

$$P_X^{(\pm)}(\tau)|_{\mathrm{class}}=\langle J^{(+)}(X)\rangle_{\pm}, \tag{51}$$

$$P_X^{(\pm)}(\tau)|_{\mathrm{quant}}=\langle\psi_{\pm}|\pm\tau;X\rangle\langle\pm\tau;X|\psi_{\pm}\rangle =\langle\psi_{\pm}(\tau)|\hat{J}^{(+)}(X)|\psi_{\pm}(\tau)\rangle. \tag{52}$$

We shall next concentrate on the operator $\hat{\mathcal{T}}(X)$. From the definition (11) one finds, taking Eq. (41) into account, that $\hat{\mathcal{T}}(X)$ satisfies

$$\pm\Theta(\pm\hat{P})\hat{\mathcal{T}}(X)\Theta(\pm\hat{P})=\Theta(\pm\hat{P})\left[\int_{-\infty}^{+\infty}d\tau\tau\hat{J}_{\mathrm{H}}^{(+)}(X,\tau)\right] \times\Theta(\pm\hat{P}), \tag{53}$$

where $\hat{J}_{\mathrm{H}}^{(+)}(X,\tau)$ denotes the positive current in the Heisenberg picture

$$\hat{J}_{\mathrm{H}}^{(+)}(X,\tau)=e^{i\hat{H}_0\tau/\hbar}\hat{J}^{(+)}(X)e^{-i\hat{H}_0\tau/\hbar}. \tag{54}$$

Accordingly, the mean arrival time at X [Eq. (23)] can be expressed as

$$\langle t_X\rangle_{\pm}=\pm\langle\psi_{\pm}|\hat{\mathcal{T}}(X)|\psi_{\pm}\rangle =\int_{-\infty}^{+\infty}d\tau\tau\langle\psi_{\pm}(\tau)|\hat{J}^{(+)}(X)|\psi_{\pm}(\tau)\rangle. \tag{55}$$

The above equation gives the mean arrival time in a form that can be recognized as a quantum version of its classical counterpart in terms of the probability distribution (45). Indeed, the positive current $\langle\psi_{\pm}(\tau)|\hat{J}^{(+)}(X)|\psi_{\pm}(\tau)\rangle$ enters Eq. (55) as a probability density of the time of arrival at X. It should be stressed that contrary to what happens with Eq. (20), by virtue of Eq. (53) the above equation is valid for *any* $|\psi_{\pm}(\tau)\rangle$ satisfying the condition (18). When in addition to this latter condition it is also satisfied that

$$\lim_{p\to\pm\infty}\langle p|\psi_{\pm}\rangle=0,\quad \lim_{p\to 0}p^{-1}\langle p|\psi_{\pm}\rangle=0, \tag{56}$$

it can be shown, after some algebra, that the mean arrival time $\langle t_X\rangle_{\pm}$ can also be expressed in the alternative forms

$$\langle t_X\rangle_{\pm}=\pm\int_{-\infty}^{+\infty}d\tau\tau\langle\psi_{\pm}(\tau)|\hat{J}(X)|\psi_{\pm}(\tau)\rangle, \tag{57}$$

$$\langle t_X\rangle_{\pm}=\left\langle\psi_{\pm}\left|\frac{1}{2}\left([X-\hat{X}_{\mathrm{H}}(0)]\frac{m}{\hat{P}_{\mathrm{H}}(0)}+\frac{m}{\hat{P}_{\mathrm{H}}(0)}[X-\hat{X}_{\mathrm{H}}(0)]\right)\right|\psi_{\pm}\right\rangle, \tag{58}$$

$$\langle t_X\rangle_{\pm}=\left\langle\psi_{\pm}\left|\sqrt{\frac{m}{\hat{P}_{\mathrm{H}}(0)}}[X-\hat{X}_{\mathrm{H}}(0)]\sqrt{\frac{m}{\hat{P}_{\mathrm{H}}(0)}}\right|\psi_{\pm}\right\rangle. \tag{59}$$

In the derivation of the above formulas use has been made of the fact that the position operator transforms under spatial translations as

$$e^{-i\hat{P}X/\hbar}\hat{X}e^{+i\hat{P}X/\hbar}=(\hat{X}-X) \tag{60}$$

and that, at $t=0$, quantum operators in the Schrödinger picture become indistinguishable from those in the Heisenberg picture, so that, in particular, we have $\hat{X}=\hat{X}_{\mathrm{H}}(0)$ and $\hat{P}=\hat{P}_{\mathrm{H}}(0)$.

Equations (58) and (59) give the mean arrival time at X in terms of the operators $\hat{T}(X)$ introduced by Grot *et al.* and by Aharonov and Bohm [Eqs. (3) and (5), respectively]. A connection between these operators and the self-adjoint ''time'' operator $\hat{\mathcal{T}}(X)$ can be derived from a comparison with Eq. (55). Indeed, as long as conditions (18) and (56) are satisfied, the expectation value of $\hat{\mathcal{T}}(X)$ coincides with the expectation values of the operators $\hat{T}(X)$, which have the interest that they can be obtained by quantizing the classical expression $t(X)=m[X-x(0)]/p(0)$ according to different standard quantization (ordering) rules.

Under the same conditions, the expression of $\langle t_X\rangle_{\pm}$ given by Eq. (55) becomes also indistinguishable from that given by Eq. (57), which involves the usual quantum probability current and has been frequently used, in practice, as a quantum definition for the mean arrival time. This fact may provide additional justification for the latter expression, whose validity in a quantum framework might, in principle, be questionable. Indeed, despite the formal analogy between Eqs. (55) and (57), they have a somewhat different physical meaning: While the positive current $\hat{J}^{(+)}(X)$ is a positive-definite operator and its expectation value enters Eq. (55) playing the role of a probability density, the expectation value of the usual probability current $\hat{J}(X)$ can take negative values and consequently cannot be interpreted as a probability distribution of arrival times. Since usually this has been the case, however, it is instructive analyzing under what circumstances the expectation values of $\hat{J}(X)$ and $\hat{J}^{(+)}(X)$ become indistinguishable. This will be the aim of the next section.

V. PROBABILITY CURRENT VERSUS POSITIVE CURRENT

In the preceding section we have seen that for state vectors satisfying Eqs. (18) and (56) the mean arrival time $\langle t_X\rangle_\pm$ can be equally calculated by using the positive current or the probability current [Eqs. (55) and (57), respectively]. Furthermore, by using

$$\int_{-\infty}^{+\infty} d\tau e^{i(p^2/2m-p'^2/2m)\tau/\hbar}=\frac{m}{|p|}\,\delta(p'-p)+\frac{m}{|p|}\,\delta(p'+p) \tag{61}$$

in Eqs. (25) and (29), we obtain

$$\int_{-\infty}^{+\infty} d\tau\langle\psi_\pm(\tau)|\hat{J}^{(+)}(X)|\psi_\pm(\tau)\rangle = \pm\int_{-\infty}^{+\infty} d\tau\langle\psi_\pm(\tau)|\hat{J}(X)|\psi_\pm(\tau)\rangle, \tag{62}$$

so that the total probability of arriving at X (at any instant of time) can also be calculated in terms of $\hat{J}^{(+)}(X)$ or $\hat{J}(X)$. Despite the interchangeable role that these quantities play in the above integral expressions, when the state vector describing the quantum particle exhibits a considerable spread in momentum space the contribution of quantum interference effects may be important and the expectation values of $\hat{J}(X)$ and $\hat{J}^{(+)}(X)$ can be appreciably different. This is most easily seen by considering a superposition of two nonoverlapping wave packets with well-defined momentum. Specifically, we shall consider a state vector $|\psi_+\rangle$ given by

$$|\psi_+\rangle=\alpha_1|\psi_1\rangle+\alpha_2|\psi_2\rangle, \tag{63}$$

where the coefficients α_1,α_2 are real and $\langle p|\psi_j\rangle$ $(j=1,2)$ are assumed to be minimum Gaussian wave packets centered at the spatial point x_0, with momentum spread Δp and average momentum p_j, respectively,

$$\langle p|\psi_j\rangle=[2\pi(\Delta p)^2]^{-1/4}\exp\left[-\left(\frac{p-p_j}{2\Delta p}\right)^2-i\frac{px_0}{\hbar}\right]. \tag{64}$$

We take $p_2>p_1>0$ and $\Delta p\ll p_2-p_1$ in order to guarantee that the above two wave packets do not overlap appreciably. Under these assumptions, the probability current $\langle\psi_+|\hat{J}_{\rm H}(X,\tau)|\psi_+\rangle$ and the probability density $\langle\psi_+|\hat{J}_{\rm H}^{(+)}(X,\tau)|\psi_+\rangle$ can be written, respectively, as

$$\langle\psi_+|\hat{J}_{\rm H}(X,\tau)|\psi_+\rangle=\frac{1}{mh}\frac{1}{2}(I^*[p]I[1]+\text{c.c.}), \tag{65}$$

$$\langle\psi_+|\hat{J}_{\rm H}^{(+)}(X,\tau)|\psi_+\rangle=\frac{1}{mh}(I^*[\sqrt{p}]I[\sqrt{p}]), \tag{66}$$

where now the functional $I[f]$ is given by $I[f]\equiv I_1[f]+I_2[f]$ with

$$I_j[f]\equiv\alpha_j[2\pi(\Delta p)^2]^{-1/4}\int_0^\infty dp\ f(p)\times\exp\left[-\left(\frac{p-p_j}{2\Delta p}\right)^2-i\frac{px_0}{\hbar}\right]\exp\left[-\frac{p^2}{2m}\frac{\tau}{\hbar}+i\frac{pX}{\hbar}\right]. \tag{67}$$

To obtain an analytical estimation (as a function of τ) for the probability current and for the probability density of arrival times [as given by Eqs. (65) and (66), respectively] we shall next consider the asymptotic expansion of the above integral. To leading order as $\Delta p\to 0$, we have [25]

$$I_j[f]\sim\alpha_j\sqrt{2}[2\pi(\Delta p)^2]^{1/4}f(p_j)\times\exp\left[-\frac{p_j^2}{2m}\frac{\tau}{\hbar}-i\ p_j(x_0-X)/\hbar\right]+O\left(\frac{(\Delta p)^3}{\sqrt{\Delta p}}\right). \tag{68}$$

By substituting Eq. (68) into Eqs. (65) and (66), one obtains, after some algebra, the asymptotic expressions

$$\langle\psi_+|\hat{J}_{\rm H}(X,\tau)|\psi_+\rangle \sim\frac{2\sqrt{2\pi}}{mh}\Delta p\{\alpha_1^2p_1+\alpha_2^2p_2+\alpha_1\alpha_2(p_1+p_2)\times\cos[(p_2^2-p_1^2)\tau/2m+(p_2-p_1)\times(x_0-X)]/\hbar+O((\Delta p)^2)\}, \tag{69}$$

$$\langle\psi_+|\hat{J}_{\rm H}^{(+)}(X,\tau)|\psi_+\rangle \sim\frac{2\sqrt{2\pi}}{mh}\Delta p\{\alpha_1^2p_1+\alpha_2^2p_2+\alpha_1\alpha_2\sqrt{p_1p_2}\times\cos[(p_2^2-p_1^2)\tau/2m+(p_2-p_1)(x_0-X)]/\hbar+O((\Delta p)^2)\}. \tag{70}$$

From Eq. (69) we see that the probability current can take negative values whenever the interference term dominates over both the first and the second one. It is not hard to see that this is the case when it holds that

$$1\ll\frac{\alpha_1}{\alpha_2}\ll\frac{p_2}{p_1} \tag{71}$$

and, under these circumstances, Eq. (69) is given, to a good approximation, by

$$\langle \psi_+ | \hat{J}_{\rm H}(X,\tau) | \psi_+ \rangle \sim \frac{2\sqrt{2\pi}}{mh} \Delta p \, \alpha_1 \alpha_2 p_2 \cos[p_2^2 \tau/2m + p_2(x_0 - X)]/\hbar + O((\Delta p)^3). \tag{72}$$

In contrast, the probability density $\langle \psi_+ | \hat{J}_{\rm H}^{(+)}(X,\tau) | \psi_+ \rangle$ remains always positive. This fact, which is evident from Eq. (66), can also be verified from the asymptotic expression (70) by noting that the dominance of the interference term would require that

$$1 \ll \frac{\alpha_1}{\alpha_2}\sqrt{\frac{p_1}{p_2}} \ll 1, \tag{73}$$

which is obviously impossible.

On the other hand, it can be readily verified that for non-normalizable states with a well-defined momentum $|\psi_\pm\rangle = |p\rangle$ $(p \neq 0)$, the quantity $\langle \psi_\pm(\tau) | \hat{J}^{(+)}(X) | \psi_\pm(\tau) \rangle$ becomes indistinguishable from the usual probability current $\pm \langle \psi_\pm(\tau) | \hat{J}(X) | \psi_\pm(\tau) \rangle$. One expects this fact to be also true for normalizable states describing particles with a highly defined momentum, and this is indeed the case as can be inferred from the asymptotic behavior (as the momentum uncertainty approaches zero) of the integrals defining the expectation values of $\hat{J}(X)$ and $\hat{J}^{(+)}(X)$ [see Eqs. (69) and (70) above and take $\alpha_2 \equiv 0$].

In scattering problems one is usually concerned with particles propagating with a well-defined velocity toward a localized interaction center. Such particles are characterized by quantum states highly concentrated in momentum space about a certain momentum $p_0 \neq 0$. Under these conditions, the expectation values of $\hat{J}(X)$ and $\hat{J}^{(+)}(X)$ coincide to a good approximation, so that the probability current yields essentially correct results for the probability density of the time of arrival at a given point. This fact provides a justification for the use of $\hat{J}(X)$ in this kind of problem.

Finally, it is worth noting that Eq. (57), which gives the mean arrival time $\langle t_X \rangle_\pm$ in terms of the probability current, is applicable even for state vectors having a large momentum uncertainty. Indeed, its validity only requires the fulfillment of conditions (18) and (56).

VI. CONCLUSION

Quantum theories are assumed to be more fundamental in nature than the corresponding classical theories. Consequently, it is possible, in principle, to define quantum quantities without resorting to the correspondence principle. In practice, however, the correspondence principle proves to be extremely useful in the construction of the quantum counterpart of a certain classical quantity. For instance, the canonical quantization method represents an invaluable tool for the construction of quantum field theories. Furthermore, one usually gets a better understanding of a quantum theory when a clear and unambiguous relationship can be established between quantum and classical quantities.

In this paper we have been particularly interested in investigating the connection between the expressions previously proposed in Ref. [7] for the probability distribution of the time of arrival at a given spatial point and their corresponding classical counterparts. In particular, we have shown that, in the semiclassical limit $\hbar \rightarrow 0$, the proposed probability density of arrival times coincides with the modulus of the quantum probability current. Indeed, Eq. (37) can be rewritten in the form

$$|\langle \pm \tau; X | \psi_\pm \rangle|^2 \sim |\langle \psi_\pm(\tau) | \hat{J}(X) | \psi_\pm(\tau) \rangle|. \tag{74}$$

This result has the interest that, at a classical level, the current of a statistical ensemble of particles propagating along a well-defined spatial direction plays the role of a probability distribution of arrival times. Therefore, Eq. (74) reflects that the quantity $|\langle \pm \tau; X | \psi_\pm \rangle|^2$ has the correct semiclassical limit and suggests that it represents a quantum version of the corresponding classical expression. We have explicitly shown that this is the case by expressing the probability distribution $|\langle \pm \tau; X | \psi_\pm \rangle|^2$ as the expectation value of a certain positive definite current operator. Indeed, by making use of the fact that the probability amplitude of arriving at X at time τ coincides with the probability amplitude of finding the state vector $\sqrt{|\hat{P}|/m}|X\rangle$ in the Schrödinger state of the particle $|\psi_\pm(\tau)\rangle$, one can write the corresponding probability density in the form

$$P_X^{(\pm)}(\tau)|_{\rm quant} \equiv |\langle \pm \tau; X | \psi_\pm \rangle|^2 = \langle \psi_\pm(\tau) | \hat{J}^{(+)}(X) | \psi_\pm(\tau) \rangle, \tag{75}$$

where the positive-definite operator

$$\hat{J}^{(+)}(X) = \sqrt{\frac{|\hat{P}|}{m}} \delta(\hat{X} - X) \sqrt{\frac{|\hat{P}|}{m}} \tag{76}$$

can be immediately recognized as a straightforward quantum version of the modulus of the classical current $|J(X)| = |p|/m \, \delta(x - X)$. The existence of a remarkable formal analogy between the corresponding classical and quantum expressions is therefore apparent: For particles propagating along a well-defined spatial direction, the probability distribution of the time of arrival at a given point can be inferred, within both a classical and a quantum framework, from the mean value of the modulus of the current.

On the other hand, for normalizable states satisfying the identity

$$|\psi_\pm(t)\rangle \equiv \Theta(\pm \hat{P}) |\psi_\pm(t)\rangle \tag{77}$$

and vanishing faster than p as p approaches zero, the mean arrival time at X, $\langle t_X \rangle_\pm$, can be equally calculated in terms of the positive-definite current $\hat{J}^{(+)}(X)$ or in terms of the standard probability current. This interchangeable role is not restricted to the mean arrival time. Indeed, we have seen that for physical states with a sufficiently well-defined momentum the expectation values of $\hat{J}(X)$ and $\hat{J}^{(+)}(X)$ become indistinguishable, so that the probability current yields essentially correct results for the probability density of the time of arrival at a given point. This fact may provide a justification for the common practice of using the expectation value of $\hat{J}(X)$ in this kind of problem.

Furthermore, under the same above conditions, the expectation value of the self-adjoint ''time'' operator $\hat{\mathcal{T}}(X)$ coincides with the expectation values of the operators $\hat{T}(X)$ introduced by Grot *et al.* and by Aharonov and Bohm [Eqs. (3) and (5), respectively], which have the interest that they can be considered as straightforward quantum versions of the classical expression $t(X)=m[X-x(0)]/p(0)$.

In summary, we have shown that the formalism developed in Ref. [7] for the time of arrival of a quantum particle at a given spatial point can be reformulated in a form that exhibits a remarkable formal analogy with the corresponding classical formulation.

Note added in proof. As stated in the Introduction, the idea that according to standard quantum mechanics measuring results of physical quantities can be inferred from the spectral decomposition of a certain self-adjoint operator, without having to make reference to the specific properties of the measuring device involved, plays a central role in our treatment. In this regard it should be mentioned that a completely different view is developed by Aharonov *et al.* [28]. These authors, by explicitly modeling the measuring device, arrive at the conclusion that the time of arrival cannot be precisely defined and measured in quantum mechanics. In the view of the present author, however, this pessimistic conclusion can be mitigated, in part, by the assumptions on which it is based [29].

On the other hand, in connection with Eq. (74) (which is only valid in the semiclassical limit) it is interesting to note that, as shown by McKinnon and Leavens [3], within the framework of Bohmian mechanics the modulus of the quantum probability current provides a definition for the probability density of arrival times which is of general applicability [30]. Therefore, while according to Eq. (75) within conventional quantum mechanics the probability density of the time of arrival can be defined as the mean value of the modulus of the current, within Bohmian mechanics it is the modulus of the mean value of the current that is the relevant quantity (the range of applicability of the two expressions is different, however).

ACKNOWLEDGMENT

This work has been supported by Gobierno Autónomo de Canarias (Project No. PI 2/95).

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